Physics 602 **Homework Set 3** **Spring 2017**

Due date: Thursday, March 23, at the end of the lecture.

Note: you are allowed to use only the textbook (Grosso & Parravicini), lecture notes, materials that I specifically provided, and Mathematica or other similar software to do the homework.

1. *Spin waves in a 2D triangular ferromagnet* A 2D ferromagnet on a triangular lattice in a magnetic field is described by the Hamiltonian

$$H = -J \sum_{j\delta} \mathbf{S}_j \cdot \mathbf{S}_{j+\delta} - B \sum_j S_j^z,$$

where j labels the lattice sites and δ denotes the nearest neighbors of site j. Assume $S \gg 1$.

1. Determine the spectrum of the spin-waves. You can use the expression we derived in class, but make sure to modify it to include B, which we did not have.
2. Compute the spin-wave contribution to the specific heat per lattice site in two limits: $T \ll B \ll J$ and $B \ll T \ll J$.

2. *Holstein-Primakoff transformation.* Prove that the spin operator as expressed in the Holstein-Primakoff transformation

$$S^+ = (2S - a^\dagger a)^{1/2} a, \quad S_- = a^\dagger (2S - a^\dagger a)^{1/2}, \quad S^z = S - a^\dagger a$$

satisfies the usual commutation relations

$$[S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm. \tag{1}$$

3. *Schwinger boson representation.* In the Schwinger boson representation, the spin operator is expressed in terms of two bosonic operators a and b

$$S^+ = a^\dagger b, \quad S^- = b^\dagger a, \quad S^z = \frac{a^\dagger a - b^\dagger b}{2}.$$

1. Show that the above expressions satisfy the usual spin commutation relations (1).
2. Show that

   $$|Sm\rangle = \frac{(a^\dagger)^{S+m} (b^\dagger)^{S-m} |0\rangle}{[(S+m)!(S-m)!]^{1/2}}$$

   is a common eigenstate of $\mathbf{S}^2$ and S_z. Note that the physical state space is given by $\{|n_a, n_b\rangle\}$ with $n_a + n_b = 2S$.

4. *Jordan-Wigner transformation.* In the Jordan-Wigner transformation spin operators living on a 1D chain are expressed as

$$S_k^+ = f_k^\dagger e^{i\pi \sum_{j<k} n_j}, \quad S_k^- = e^{-i\pi \sum_{j<k} n_j} f_k, \quad S_k^z = f_k^\dagger f_k - \frac{1}{2},$$

where f_k and $f_k^\dagger$ are the annihilation and creation operators of spineless fermions and $n_j = f_j^\dagger f_j$ as usual.

1. Show that
$$S_k^+ S_{k+1}^- = f_k^\dagger f_{k+1}.$$

2. The Hamiltonian of an anisotropic quantum Heisenberg spin-1/2 chain is given by
$$H = -\sum_k \left[\frac{1}{2} J_\perp (S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+) + J_z S_k^z S_{k+1}^z \right].$$
Show that the Hamiltonian can be written as
$$H = -\sum_k \left[\frac{1}{2} J_\perp (f_k^\dagger f_{k+1} + f_{k+1}^\dagger f_k) + J_z \left(\frac{1}{4} - f_k^\dagger f_k + f_k^\dagger f_k f_{k+1}^\dagger f_{k+1} \right) \right].$$

3. Show that for 1D spin-1/2 XY-model, i.e. $J_z = 0$, the eigenvalues of the Hamiltonian H are given by
$$\hbar\omega_k = -J_\perp \cos(ka),$$
where a is the lattice spacing.