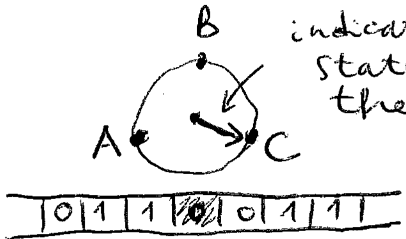
Maxwell's demon

Chris Jarzynski $\Rightarrow$ explicit mechanistic model of a Maxwell's demon



indicates the state of the

$A \to B \to C \to A$ CCW rotations

$A \leftarrow B \leftarrow C \leftarrow A$ CW

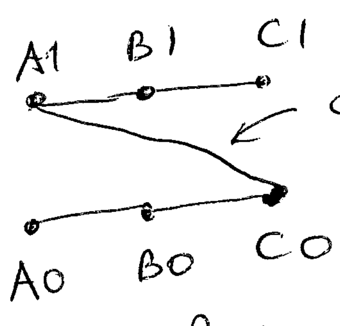
interaction time: τ $\rightarrow$ frictionless tape, bits move at constant speed

$\tau^{-1} \Leftarrow$ rate at which bits pass the demon (& interact with it)

Introduce X: $\begin{cases} X \to X+1 & \text{for each } C \to A \\ X \to X-1 & \text{for each } A \to C \end{cases}$

Demon + bit: $\underbrace{A0,\ B0,\ C0,\ A1,\ B1,\ C1}_{6 \text{ states}}$

Transitions between states:



coupled

demon-bit transition

linear chain of states

Rates R_{ij} : prob. of transition per unit time

$R_{ij} = R_{ji} = 1$ ← sets the unit of time

Relaxation occurs for times τ_r of $O(1)$

$$R \approx \begin{array}{cccccc} A0 & B0 & C0 & A1 & B1 & C1 \end{array}$$
$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ +1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$\tau \ll \tau_r = \mathcal{O}(1)$ no relaxation

$\tau \gg \tau_r$ demon/bit system reaches equil.

Define: $b_n = \{0,1\}$ incoming bit

$b_n' = \{0,1\}$ outgoing bit

Incoming bits: $p_0 \Rightarrow$ prob. to be $\emptyset$

$p_1 = 1 - p_0 \Rightarrow$ prob. to be 1

$\delta = p_0 - p_1$ excess prm

Imagine that all incoming bits are $\emptyset$: start in $A\emptyset$, $B\emptyset$, or $C\emptyset$

During τ: may have multiple transitions $C0 \to A1$ & $A1 \to C0$.

May have $\Delta X_n \equiv X(t_{n+1}) - X(t_n) = 0$

$t_{n+1} = t_n + \tau$

System ends up in $A0, B0, C0$

or $\Delta X_n = +1 \Rightarrow$ net rotation (!)

(if $\emptyset$ replaced by 1)

system ends up in $A1, B1, C1$

As the next bit comes in
(say the system is ~~in~~ in B1):

$B1 \to B0$ "transition"
(bit replacement)

Repeat with the next bit...

$\Delta X = \sum_n \Delta X_n$ increases with time
(never decreases when all incoming bits are $\emptyset$) $\Rightarrow$

$\Rightarrow$ net rotation

Record of rotation events in the outgoing bit stream...

Same, but with the rotation in the opposite direction, when all incoming bits are 1.

In general, $\Delta X_n = b_n' - b_n$ &
net rotation may be produced by ~~[illegible]~~ "biased" bit streams.

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \Rightarrow \begin{pmatrix} p_0' \\ p_1' \end{pmatrix}$$

incoming outgoing

$$\delta = p_0 - p_1 \Rightarrow \delta' = p_0' - p_1'$$

$$\varphi \equiv \langle \underset{x'''-x^{\circ}}{\Delta X} \rangle = p_1' - p_1 = \frac{\delta - \delta'}{2}$$

$$\langle x \rangle = p_1 \cdot 1 + p_0 \cdot 0 = p_1$$

indeed,
$$\frac{p_0 - p_1 - p_0' + p_1'}{2} = \frac{1 - 2p_1 - 1 + 2p_1'}{2} = p_1' - p_1$$

kmon reaches periodic steady state eventually:

define $\mathcal{T}_{3\times3}$ – transition matrix between states A, B, C :

$$\mathcal{T} = \underbrace{\left(\begin{array}{ccc|ccc} 1 & & 0 & 1 & & 0 \\ 0 & 1 & & 0 & 1 & \\ & & 1 & & & 1 \end{array}\right)}_{3\times6} \underbrace{e^{L\tau}}_{6\times6} \underbrace{\begin{pmatrix} p_0 & & 0 \\ 0 & p_0 & \\ & & p_0 \\ p_1 & & 0 \\ 0 & p_1 & \\ & & p_1 \end{pmatrix}}_{6\times3}$$

For example, if $\tau \to 0$,

$$\mathcal{T} = \begin{pmatrix} p_0+p_1 & & 0 \\ & p_0+p_1 & \\ 0 & & p_0+p_1 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

as expected

In steady state (ss):

$$\mathcal{T} \underset{\begin{pmatrix} q^A \\ q^B \\ q^C \end{pmatrix}}{\underset{\parallel}{q^{pss}}} = q^{pss}$$

Finally, we can find the stats of the outgoing bit:

$$\begin{pmatrix} p_0' \\ p_1' \end{pmatrix} = \underbrace{\begin{pmatrix} 111 & 000 \\ 000 & 111 \end{pmatrix}}_{2\times 6} \underbrace{e^{R\tau}}_{6\times 6} \underbrace{\begin{pmatrix} p_0 & & 0 \\ & p_0 & \\ 0 & & p_0 \\ \hline p_1 & & 0 \\ & p_1 & \\ 0 & & p_1 \end{pmatrix}}_{6\times 3} \begin{pmatrix} q^A \\ q^B \\ q^C \end{pmatrix}$$

Note that

$$\underbrace{\begin{pmatrix} p_0 & & \\ & p_0 & \\ & & p_0 \\ p_1 & & \\ & p_1 & \\ & & p_1 \end{pmatrix}}_{6\times 3} \underbrace{\begin{pmatrix} q^A \\ q^B \\ q^C \end{pmatrix}}_{3} = \underbrace{\begin{pmatrix} p_0 q^A \\ p_0 q^B \\ p_0 q^C \\ p_1 q^A \\ p_1 q^B \\ p_1 q^C \end{pmatrix}}_{6} \Leftarrow \text{combined state of the demon-bit system}$$

if $\tau \to 0$:

$$\begin{pmatrix} p_0' \\ p_1' \end{pmatrix} = \begin{pmatrix} 111 & 000 \\ 000 & 111 \end{pmatrix} \begin{pmatrix} p_0 q^A \\ p_0 q^B \\ p_0 q^C \\ p_1 q^A \\ p_1 q^B \\ p_1 q^C \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix},$$

as expected

One can find $e^{R\tau} \Rightarrow$

$$\Rightarrow T_0 \Rightarrow q^{pss} \Rightarrow \begin{pmatrix} p_0' \\ p_1' \end{pmatrix} \Rightarrow \varphi$$

It turns out that

$$\varphi(\delta, \tau) = \frac{\delta}{2}\left[1 - \frac{1}{3} K(\tau)\right]$$

$$\lim_{\tau \to 0} K(\tau) = 3 \Rightarrow \varphi = 0$$

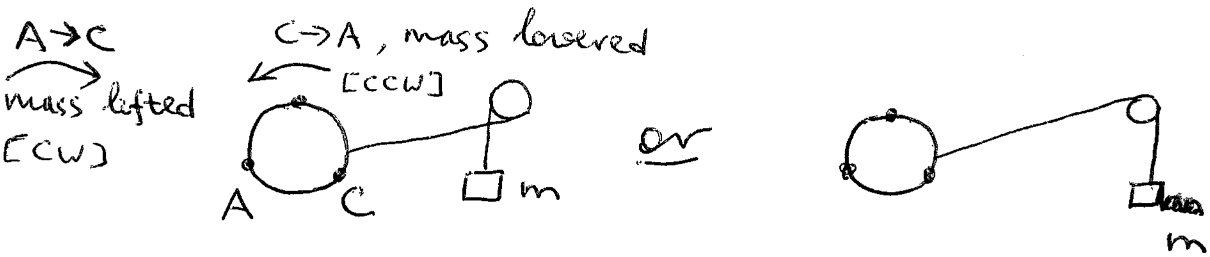
$$K(\tau) \sim e^{-2\tau}$$

$$\lim_{\tau\to\infty} K(\tau) = 0 \Rightarrow \varphi = \frac{\delta}{2}$$

$\Uparrow$

System relaxes to equil., all 6 states equally likely $\Rightarrow \delta' = 0 \Rightarrow$
$\Rightarrow \varphi = \frac{\delta}{2}$.

Now add the load:



Additional (funny) simplification: <u>only</u> C→A transitions change the energy (by $\pm mg\Delta h$). This energy is exchanged with the heat bath.

So, now A1, B1, C1 have different energy from A0, B0, C0:

$$\frac{R_{A1,C0}}{R_{C0,A1}} = e^{-\underbrace{\frac{mg\Delta h}{k_B T}}_{f>0}}$$

At equil., $\begin{cases} p_{A0}^{eq} = p_{B0}^{eq} = p_{C0}^{eq} = \frac{e^f}{Z}, \\ p_{A1}^{eq} = p_{B1}^{eq} = p_{C1}^{eq} = \frac{1}{Z}, \end{cases}$

where $Z = 3(1+e^f)$.

Further,

$$p_0^{eq} - p_1^{eq} = \sum_{i \in \{A,B,C\}} \left(p_{i0}^{eq} - p_{i1}^{eq}\right) = \frac{e^f - 1}{e^f + 1} = \tanh\left(\frac{f}{2}\right) \equiv \xi .$$

Note that $\mathrm{sgn}(\xi) = \mathrm{sgn}(f)$.

We can choose $\begin{cases} R_{A1,C0} = 1 - \xi, \\ R_{C0,A1} = 1 + \xi . \end{cases}$

Indeed, $\dfrac{R_{A1,C0}}{R_{C0,A1}} = \dfrac{1-\xi}{1+\xi} = e^{-f}$, consistent w/above

So, now we have:

$$R = \begin{array}{c} \begin{array}{cccccc} A0 & B0 & C0 & A1 & B1 & C1 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2+\xi & 1+\xi & 0 & 0 \\ 0 & 0 & 1-\xi & -2-\xi & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

We can obtain:

$$\varphi(\delta, \xi; \tau) = \frac{\delta - \xi}{2}\left[1 - \frac{1}{3}K(\tau) + \frac{\xi\delta}{6} J(\tau, \xi\delta)\right]$$

$\xi \to 0$ recovers the previous expression

$\tau \to \infty$: $J(\tau, \xi\delta) \sim e^{-4\tau} \to 0 \Rightarrow \varphi \to \dfrac{\delta - \xi}{2}$

$\tau \to 0$: $J(\tau, \xi\delta) \to 0 \quad \Rightarrow \varphi \to 0$ (no change)

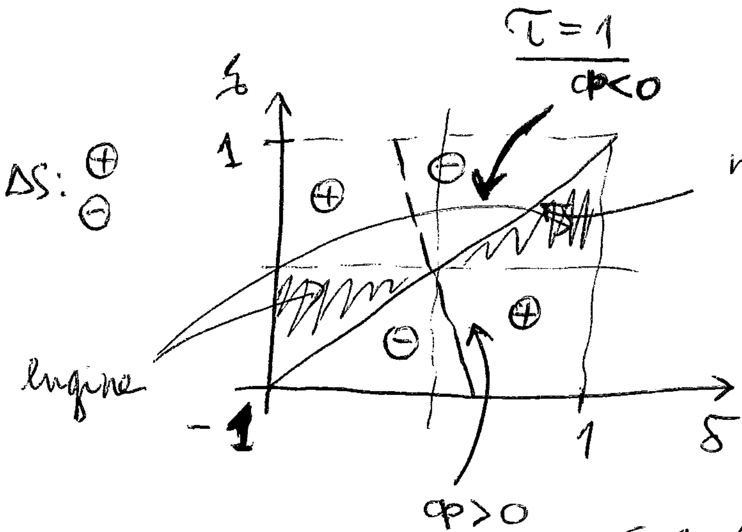
demon-bit equilibration at each step: $p'_{0,1} = p_{0,1}^{eq} \Rightarrow \delta' = p_0^{eq} - p_1^{eq} = \xi$,

so that $\varphi = \dfrac{\delta - \xi}{2}$

$\delta = \xi_b$: no net rotation, incoming bits already @ equil. Otherwise net rotation observed.

Note that $|\delta| \le 1$, $\underbrace{|\xi_b| \le 1}_{\text{weight hangs off the right or left side of the pulley}}$, $0 < \tau < \infty$.

"Phase diagram":



no net rotation at steady state

$$W = \underbrace{k_B T f}_{mg\Delta h} \phi \leftarrow \text{average work per interaction interval}$$

$\text{sign}(f) = \text{sign}(\xi_b) \Rightarrow$

$\Rightarrow \xi_b \phi > 0$ regions are $W > 0$ regions, where the device acts as an engine (converts heat from the thermal bath into mechanical work)

Now define
$$\begin{cases} S_b = -\sum_{i=0,1} p_i \log p_i \\ S_b' = -\sum_{i=0,1} p_i' \log p_i' \end{cases}$$
"disorder per bit"

[ignores correlations between bits]

S_b or S_b' range from $\emptyset$ [stream of all $\emptyset$'s or 1's] to $\log 2$.

Consider $\Delta S = S_b' - S_b$ characterizes the change in bit distribution after interacting w/the demon

The $\delta_1 = \delta$ line ($\varphi = 0$) gives $\Delta S = 0$: no net rotation, $\begin{cases} p_0' = p_0 \\ p_1' = p_1 \end{cases}$ "trivial"

However, $\underset{+p_0 \log p_0 + p_1 \log p_1 - p_1 \log p_1 - p_0 \log p_0}{\Delta S \overset{\shortparallel}{=} 0}$ if $\begin{cases} p_0' = p_1 \\ p_1' = p_0 \end{cases}$ "flipped"

Note that $\Delta S > 0$ whenever the demon acts as an engine ($W > 0$) $\Rightarrow$ the demon converts heat to work & writes info to the memory register. Alternatively, "evolution" towards a more disordered sequ of 0's & 1's can drive a thermodynamic engine.

In $\Delta S < 0$ regions, the demon removes disorder from the bit stream: e.g. if $\delta = 0$ (50/50 mixture of 0's & 1's), and if $\nu f \gg 1 \Rightarrow \epsilon \approx 1$, $\tau \gg 1$: the demon equilibrates to $\underbrace{A0, B0, C0}_{\text{much more favorable in energy}}$ $\Rightarrow$ outgoing bits are almost all 0's $\Rightarrow$ memory "wiped clean", $W < 0$ (device consumes energy)

In $\Delta S > 0$, $W < 0$ regions, entropy of the bit stream increases & the energy is used up as well $\Rightarrow$ no useful f'n

Next, it turns out that $W \le k_B T \Delta S$
$[W = k_B T \Delta S$ iff $\epsilon = \delta]$

Indeed,

$$W = k_B T f \varphi = k_B T \varphi \log \frac{1+\epsilon}{1-\epsilon}.$$

Consider $\underset{\substack{\uparrow \\ \text{dissipation f'n}}}{\Omega} \equiv \Delta S - \varphi \log \frac{1+\epsilon}{1-\epsilon}$.

For simplicity, focus on the $\tau \to \infty$ case.

In this limit, $\delta' \to \epsilon$; $\varphi \to \frac{\delta - \epsilon}{2}$;
$\Delta S \to S(\epsilon) - S(\delta)$,

hence $$\Omega \to S(\epsilon) - S(\delta) - \frac{\delta-\epsilon}{2} \log \frac{1+\epsilon}{1-\epsilon} \equiv \Omega_\infty$$

Note that $\Omega_\infty = 0$ for $\epsilon = \delta$;

$$\frac{\partial \Omega_\infty}{\partial \epsilon} = \frac{1}{2}\log\frac{1-\epsilon}{2} - \frac{1}{2}\log\frac{1+\epsilon}{2} - \left(-\frac{1}{2}\right) - \frac{1}{2} + \frac{1}{2}\log\frac{1+\epsilon}{1-\epsilon} - \frac{\delta-\epsilon}{2}\,\frac{1-\epsilon}{1+\epsilon}\left[\frac{1}{1-\epsilon} + \frac{1+\epsilon}{(1-\epsilon)^2}\right]$$

$$S(\epsilon) = -\underbrace{\frac{1-\epsilon}{2}}_{p_0}\log\frac{1-\epsilon}{2} - \underbrace{\frac{1+\epsilon}{2}}_{p_1}\log\frac{1+\epsilon}{2}$$

(11)

$p_0 - p_1 = \epsilon$, as before

$$\Rightarrow -\frac{\delta-\epsilon}{2(1+\epsilon)}\left[1+\frac{1+\epsilon}{1-\epsilon}\right] = -\frac{\delta-\epsilon}{\cancel{2}(1+\epsilon)}\,\frac{\cancel{2}}{1-\epsilon} = \frac{\epsilon-\delta}{1-\epsilon^2} =$$

$$= \begin{cases} >0, & \epsilon>\delta \\ <0, & \epsilon<\delta \end{cases} \Rightarrow \underline{\Lambda_\infty \geq 0} \quad [?]$$

So, $\underline{W \leq k_B T \Delta S}$

if $\underset{S_b' - S_b}{\underset{\shortparallel}{\Delta S}} < 0$ ("eraser") $\Rightarrow$ $W<0$, work must be supplied

If $S_b' = 0$ ("full eraser") $\Rightarrow$

$\Rightarrow$ recover Landauer's principle,

$$|W| > k_B T S_b$$

2nd law of thermodynamics:

$$-\Delta S_r = \frac{W}{k_B T} \Leftarrow \text{decrease in reservoir's entropy}$$

$$\Downarrow$$

$\Delta S_r + \Delta S \geq 0$. entropy increases