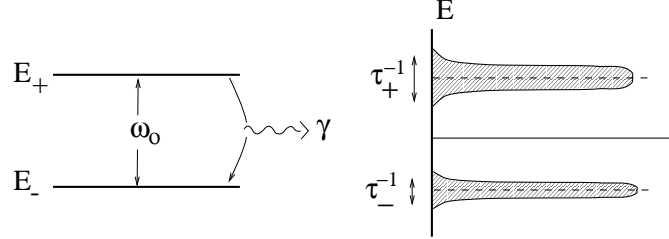




2. A simple model an atom with two atomic levels coupled to a radiation field is described by the Hamiltonian

$$H = H_o + H_I + H_{\text{photon}}, \quad (5)$$



where

$$H_o = \tilde{E}_- c_-^\dagger c_- + \tilde{E}_+ c_+^\dagger c_+ \quad (6)$$

describes the atom, treating it as a *fermion*

$$H_I = V^{-1/2} \sum_{\vec{q}} g(\omega_{\vec{q}}) \left( c_+^\dagger c_- + c_+^\dagger c_- \right) \left[ a_{\vec{q}}^\dagger + a_{-\vec{q}} \right] \quad (7)$$

describes the coupling to the radiation field ( $V$  is the volume of the box enclosing the radiation) and

$$H_{\text{photon}} = \sum_{\vec{q}} \omega_{\vec{q}} a_{\vec{q}}^\dagger a_{\vec{q}}, \quad (\omega_{\vec{q}} = cq) \quad (8)$$

is the Hamiltonian for the electromagnetic field. The “dipole” matrix element  $g(\omega)$  is weak enough to be treated by second order perturbation theory and the polarization of the photon is ignored.

- (i) Calculate the self-energy  $\Sigma_+(\omega)$  and  $\Sigma_-(\omega)$  for an atom in the  $+$  and  $-$  states.  
(ii) Use the self-energy obtained above to calculate the life-times  $\tau_\pm$  of the atomic states, i.e.

$$\tau_\pm^{-1} = 2\text{Im}\Sigma_\pm(\tilde{E}_\pm - i\delta). \quad (9)$$

If the gas of atoms is non-degenerate, i.e the Fermi functions are all small compared with unity,  $f(E_\pm) \sim 0$  show that

$$\begin{aligned} \tau_+^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o) [1 + n(\omega_o)] \\ \tau_-^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o) n(\omega_o), \end{aligned} \quad (10)$$

where  $\omega_o = \tilde{E}_+ - \tilde{E}_-$  is the separation of the atomic levels and

$$F(\omega) = \int \frac{d^3q}{(2\pi)^3} \delta(\omega - \omega_q) = \frac{\omega^2}{2\pi c^3} \quad (11)$$

is the density of state of the photons at energy  $\omega$ . What do these results have to do with stimulated emission? Do your final results depend on the initial assumption that the atoms were fermions?

- (iii) Why is the decay rate of the upper state larger than the decay rate of the lower state by the factor  $[1 + n(\omega_o)]/n(\omega_o)$ ?