Modern Optics Final Exam Solutions

December 23, 2002

Answer any twelve of the fifteen questions below in your bluebook. Partial credit will be given: show all of your calculations and your line of reasoning clearly. You can use a calculator and two sheets of formulas, but no other books or notes.

Possibly useful formulas:

\[
\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)
\]

\[
\sin(\alpha) - \sin(\beta) = 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right)
\]

\[
\int_{0}^{\infty} dx \frac{\sin(x)}{x} = \frac{\pi}{2}
\]

\[
\int_{-\infty}^{\infty} dx \exp(-ax^2 + bx) = \sqrt{\frac{\pi}{a}} \exp\left(b^2/(4a)\right)
\]

(1) Describe how you could determine whether or not the light emitted by a laser pointer is linearly polarized if you have a slab of glass (n=1.5) and a white wall available.

The polarization angle (Brewster’s angle) for \( n = 1.5 \) is \( \arctan(1.5) = 56^\circ \). For incident light with \( \theta_i \) near \( 56^\circ \) linearly polarized light polarized perpendicular to the plane of incidence will be reflected much more strongly than light polarized in the plane of incidence. (See Hecht, Fig. 4.48.) Thus if one holds the laser such that its beam is incident on the glass at an angle near \( 56^\circ \) and rotates it about its axis of symmetry, the intensity of the reflected beam, as seen by the brightness of a spot on the wall, will change from maximum to minimum every \( 90^\circ \) of rotation. If the emitted light were unpolarized there would be no change during the rotation.
(2) A plane wave of wavelength $\lambda = 550 \text{ nm}$ is incident normally on an opaque screen with three narrow parallel slits separated by distances $a = 2.3 \text{ mm}$. An interference pattern is observed on the other side of the screen at a large distance from it.

(a) At what angles are the first principal maxima adjacent to the central maximum?

(b) How does the intensity $I_{\text{max}}$ at the principal maxima compare to that from a single slit $I_1$?

(c) How does the intensity $I_h$ at angles halfway between the principal maxima compare to that from a single slit?

(a) The primary maxima occur when $a \sin(\theta) = m \lambda$, $m = 0, \pm 1, \pm 2, \ldots$ so the $m = 1$ maximum has $\sin(\theta_1) \approx \theta_1 = \lambda/a = 550 \text{ nm} / 2.3 \text{ mm} = 2.39 \times 10^{-4}$ or 0.014°.

(b) Since light from the 3 slits is in phase $E_{\text{max}} = 3E_1$ and $I_{\text{max}} = 9I_1$.

(c) At $\theta = \theta_1/2$ the path length difference between adjacent slits is $\lambda/2$ so the light from the central slit is exactly out of phase with that from the outer slits. Thus $E_h = E_1 - E_1 + E_1 = E_1$ and $I_h = I_1$.

(3) A half-wave plate is placed between two polarizers with parallel transmission axes. Its fast axis is at an angle $\theta$ relative to the transmission axes of the polarizers. Unpolarized light of intensity $I_0$ is incident on this system. Make a labelled plot of the intensity $I_f$ of the light emerging from the system, after passing through all 3 components, as a function of the angle $\theta$ for $0^\circ \leq \theta \leq 180^\circ$.

The half-wave plate rotates linearly polarized light with its polarization at an angle $\theta$ relative to its fast axis to an angle $\theta$ on the other side of this axis for a net rotation of $2\theta$. The first polarizer gives linearly polarized light of half the incident intensity, so using Malus’s Law $I_f = (I_0/2) \cos(2\theta)^2$, which is plotted below.

(4) The Fraunhofer diffraction pattern due to a plane wave of irradiance $I_{\text{inc}}$ normally incident on an $a \times b$ rectangular aperture is given by

$$I(Y, Z) = I(0)(\sin(\alpha')/\alpha')^2(\sin(\beta')/\beta')^2;$$

where $\alpha' = kaZ/(2R)$ and $\beta' = kbY/(2R)$, with $R$ the distance of the observing screen from the aperture and $Y$ and $Z$ the coordinates of the point on the observing screen.

(a) How does the total rate at which energy falls on the observing screen compare to the rate at which energy passes through the aperture?

(b) Use your answer to (a) to express $I(0)$ in terms of $I_{\text{inc}}$.

(a) By conservation of energy the two must be equal.
(b) The total energy falling on the observing screen is just the integral over all \( Y \) and all \( Z \) of \( I(Y, Z) \). Changing the variables of the two integrations to \( \alpha' \) and \( \beta' \) and using the definite integral on the first page, the equality of the two energy rates then gives

\[
I_{\text{inc}}ab = (R\lambda)^2 I(0)/(ab),
\]

or

\[
I(0) = I_{\text{inc}}(ab/(R\lambda))^2.
\]

The dependence of the variables can be easily understood since the diffraction pattern on the observing screen is more spread out the larger \( R \) and \( \lambda \) and the smaller \( a \) and \( b \). The more spread out the pattern the lower must be the intensity at any point, in particular \( I(0) \).

(5) The graph on the last page of the exam shows the intensity of light on the axis behind an opaque screen with a circular aperture of radius 1.2 mm as a function of distance \( r_0 \) from the screen when light of wavelength 510 nm is incident.

(a) What is the distance of the point A where the intensity is zero from the screen?
(b) How does the maximum intensity \( I_{\text{max}} \) in the graph compare to the unobstructed intensity \( I_{\text{un}} \) (for the case of no opaque screen at all)?

(a) This problem is most simply approached using Fresnel zones. The aperture contains exactly \( n \) Fresnel zones if

\[
r_0 = r_{0,n} = a^2/(n\lambda) = (1.2\,\text{mm})^2/(n\times510\,\text{nm}) = 2.82\,\text{m}/n.
\]

The intensity will vanish if \( n = 2, 4, 6, \ldots \) since with an even number of zones the successive zones have opposite signs and cancel. The point A is the largest of these zeroes and thus its distance is \( r_{0,2} = 1.41\,\text{m} \).
(b) The intensity is maximum when \( n \) is odd since then the contribution from one full zone survives. This is twice the amplitude for the unobstructed case, and thus the maximum intensity in the graph must be 4 times the unobstructed intensity.

(6) Design a beam expander to convert a beam of parallel rays into a beam of parallel rays of three times the diameter. Assume you have available a diverging lens of focal length -10 cm with a diameter larger than that of the incident beam and a large variety of converging lenses.
(a) What focal length should you choose for the converging lens?
(b) How far should it be placed from the diverging lens?
(c) How will the intensity of the final expanded beam compare to that of the incident beam?
(a) After passing through the diverging lens the beam of parallel rays will diverge as if coming from a virtual point source 10 cm to the left of the diverging lens. If this point source is at the focus of a converging lens the diverging rays will be converted again into parallel rays. To produce an increase in diameter by a factor of 3 the distance from the virtual point source to the converging lens must be 30 cm from the source (by similar triangles) and thus the converging lens must have a focal length of 30 cm.

(b) From the above argument it must be 30 cm - 10 cm = 20 cm to the right of the diverging lens.

(c) By conservation of energy the intensities times the areas must be the same, so the intensity in the expanded beam is 1/9 that of the original beam.

(7) Light with a mean wavelength $\lambda = 550nm$ and line width $\Delta \lambda = 5nm$ is used in Young’s Experiment with a slit separation of 2mm. The diffraction pattern is viewed on a screen 0.8 m from the slits. About how many interference maxima will be clearly visible on the screen? (Assume the slits are so narrow that variations due to the single-slit pattern can be ignored.)

The nth maximum is at a point whose distances from the two slits differs by exactly $n$ wavelengths. If this difference in path length is larger than the coherence length $\Delta l_c = c\Delta t_c = c/\Delta \nu$ the light from the two slits will be incoherent at the screen and thus no interference effects are possible. Since $\nu = c\lambda$, $\Delta \nu = c\Delta \lambda/\lambda^2$ and the fringes will disappear if $n\lambda \geq \Delta l_c = \lambda^2/\Delta \lambda$, or when $n \geq \lambda/\Delta \lambda = 550nm/5nm = 110$. These appear on both sides of the central maximum so about 221 fringes will be able to be seen clearly. (One should check that the angles involved are not greater than 90 degrees.)

(8) Consider a two-level atom with lower level $i$ and upper level $j$ with energies $E_i$ and $E_j$. Assume a total of $N = N_i + N_j$ atoms, with $N_i$ in the lower level and $N_j$ in the upper level. Then

$$dN_i/dt = -dN_j/dt = -BN_iu_\nu + BN_ju_\nu + AN_j,$$

where $A$ and $B$ are the Einstein coefficients.

(a) Identify the three terms on the right side of this equation.

(b) What is $u_\nu$?

(c) Explain why a population inversion must be produced before a laser can operate.

(a) The first term $-BN_iu_\nu$ represents (stimulated) absorption on the lower level: a photon in the beam is absorbed and the atom makes a transition from the lower to the upper level. The second term $+BN_ju_\nu$ represents stimulated emission from the upper level: the incident photons stimulate the atom to make a transition from the upper level to the lower, emitting a new photon. The third term $AN_j$ represents spontaneous emission: the atom in the excited state...
makes a transition to the lower state, emitting a photon. This takes place independent of the number of photons in the incident beam, in particular even if there are none. All photons discussed of course have energy equal to the difference between the two atomic levels.

(b) $u_{\nu}$ is the *spectral* energy density, i.e. the energy density per frequency interval. (c) A laser has to amplify the beam of photons passing through it so there must be a population inversion: $N_j$ must be larger than the value which would make the right hand side of the equation vanish, so that $dN_j/dt$ in the equation is negative and more photons are produced. (Of course pumping must be used to produce or maintain this inversion: this is not included in the equation.)

(9) Natural (unpolarized) light travelling in the $+z$ directions is scattered by milky water in a transparent container centered at the origin.

(a) An observer looks at the container from the $+x$ axis and observes scattered light. Is this scattered light polarized and, if so, in what direction?

(b) Suppose the incident light passes through a polarizer with its transmission axis in the $x$ direction before reaching the container. Describe any changes seen by the observer on the $+x$ axis.

(a) The incident light contains any polarization in the $x$-$y$ plane and thus induces dipoles in milky water in this plane which can always be decomposed into dipoles in the $x$ direction and dipoles in the $y$ direction. The former do not radiate along the $x$ axis, while the latter emit radiation along the $x$ axis which is in the $y$ direction. Thus the observer on the $+x$ axis will see scattered light polarized in the $y$ direction.

(b) In this case only dipoles in the $x$ direction will be induced in the milky water and these do not radiate in the $x$ direction. In this case the observer will see no scattered radiation.

(10) When blue ink from a felt marker pen of the kind used to make transparencies for use on an overhead projector is smeared heavily on a black surface and allowed to dry it strongly reflects red light. Explain this phenomenon. Hint: For a complex index of refraction the reflectance is given by $|n_1/(n+1)|^2$.

The ink produces blue light from a transparency because it absorbs red light more strongly than blue, so that mainly blue light is transmitted. When the ink dries on the black object it is seen mainly by light reflected from its surface. Since the concentrated ink absorbs red light very strongly (the complex index of refraction has a large imaginary part for red light, not so large for blue) it is a better reflector for red light than for blue and thus the spot of dried ink appears red.

(11) An electromagnetic standing wave can be formed from the superposition of two plane waves of the same frequency $\omega$ and amplitude $E_0$ travelling in
opposite directions. For example

\[ E_y(x, t) = E_{0y} \sin(kx - \omega t) + E_{0y} \sin(kx + \omega t) = 2E_{0y} \sin(kx) \cos(\omega t). \]

(a) What is the magnetic field \( B \) in this standing wave?
(b) What is the time average of the total energy density in this standing wave?
(c) What is the time average of the Poynting vector in this standing wave?

(a) From Maxwell’s equations the \( B \)'s for the two plane waves must have magnitudes \( E_{0y}/c \) and be in the \( z \) direction, but in opposite directions. (The simplest way to see this is from the requirement that the Poynting vectors must be in the directions in which the waves travel.) Thus

\[ B_z = (E_{0z}/c) \sin(kx - \omega t) - (E_{0z}/c) \sin(kx + \omega t) = 2(E_{0y}/c) \cos(kx) \sin(\omega t) \]

(assuming the trigonometric identity on the first page).
(b) Using

\[ u = u_E + u_B = (\epsilon_0/2)E_y^2 + (1/2\mu_0)B_z^2 \]

and \( c^2 = 1/(\epsilon_0\mu_0) \) this simplifies to

\[ u = 2\epsilon_0E_{0y}^2 [\sin(kx)^2 \cos(\omega T)^2 + \cos(kx)^2 \sin(\omega t)^2] \]

. Taking the time average replaces the \( \cos(\omega T)^2 \) and \( \sin(\omega t)^2 \) each by \( 1/2 \) and then

\[ < u > = \epsilon_0E_{0y}^2 [\sin(kx)^2 + \cos(kx)^2] = \epsilon_0E_{0y}^2 \]

for any \( x \).
(c) Since the Poynting vector is proportional to the cross product of \( E \) and \( B \) its time dependence is \( 2 \sin(\omega t) \cos(\omega t) = \sin(2\omega t) \), the time average of which is zero. This is as expected since the standing wave averaged over time does not carry energy in one direction or the other: if it did energy would build up somewhere.

(12) The spectral energy density in a blackbody cavity at temperature \( T \) is

\[ \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}. \]

Describe how you would use this expression to find the number of photons per volume in the cavity in with frequencies in the visible region between \( \nu_1 = 3.84 \times 10^{14} \text{Hz} \) and \( \nu_2 = 7.69 \times 10^{14} \text{Hz} \). (It is not necessary to evaluate this number: just describe how it could be done.)

Since each photon associated with a frequency \( \nu \) has an energy \( h\nu \) the number of photons per volume per energy per frequency interval is just the given spectral
energy density divided by $h\nu$. The number of photons per volume in the visible region is then just the integral of this between $\nu_1$ and $\nu_2$:

$$n_{\text{visible}} = \int_{\nu_1}^{\nu_2} \frac{8\pi\nu^2}{e^3} \frac{1}{e^{h\nu/k_B T} - 1}.$$  

(13) Suppose you want to design a quarter-wave plate using calcite which has $n_o = 1.6584$ and $n_e = 1.4864$.
(a) How should the optic axis be aligned?
(b) How thick should the plate be for use with light with wavelength 550 nm?
(c) If you want to use the plate to produce circularly polarized light from linearly polarized light how should the linear polarization be oriented?

(a) The optic axis should be parallel to the face of the plate.
(b) There should be a phase difference of $\pi/2$ so

$$(2\pi/\lambda_0)\Delta n d = \pi/2$$

or

$$d = \lambda_0/(4\Delta N) = (1/4)550nm/(1.6584 - 1.4864) = 799nm.$$  

(Phase differences of $2\pi m + \pi/2$, giving $d = (4m + 1) \times 799nm$, where $m = 0, 1, 2, ...$ would also work )

(c) The linear polarization should be at 45 degrees to the optic axis so that

$$\hat{i} + \hat{j} \rightarrow \hat{i} \pm \hat{j},$$

resulting in circularly polarized light.

(14) A square pulse of duration $\tau$ centered at time $t_0$ is described by the function

$$f(t) = \begin{cases} 0, & \text{if } |t - t_0| > \tau/2; \\ E_0, & \text{if } |t - t_0| < \tau/2. \end{cases}$$

(a) Find its Fourier transform $F(\omega)$.
(b) Suppose the width $\Delta \omega$ of $F(\omega)$ is defined to be the smallest value of $|\omega|$ for which $F(\omega)$ vanishes. If $\tau = 2ns$ and $t_0 = 3.7ms$ what is $\Delta \omega$

(a)

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) = E_0 \int_{t_0-\tau/2}^{t_0+\tau/2} e^{-i\omega t} = (2E_0/\omega)e^{-i\omega t_0} \sin(\omega\tau/2).$$

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(b) $F(\omega)$ does not vanish at $\omega = 0$ so its first zero is when $|\omega|\tau/2 = \pi$, or $\Delta\omega = 2\pi/\tau = 3.14\text{GHz}$ for $\tau = 2\text{ns}$, independent of the value of $t_0$.

(15) A light wave described by

$$E_{\text{inc}} = E_0 \cos(kx - \omega t)$$

in the region $x < 0$ is normally incident on a transparent dielectric slab of thickness $d$ and index of refraction $n$ occupying the region between the $x = 0$ and $x = d$ planes.

(a) Write the formula for the transmitted wave $E_t$ in the region $x > d$. Define or explain any symbols which you introduce.

(b) One can write $E_t = E_{\text{inc}} + E'$. Explain the origin of $E'$

(a) While passing through the slab the wave picks up an extra phase of $\phi = (n - 1)kd$ and so the transmitted wave after the slab is

$$E_t = t t' \cos(kx - \omega t + \phi)$$

where $t$ and $t'$ are the amplitude transmission coefficients at the two surfaces of the slab.

(b) $E'$ is the electric field due to all of the charges in the slab which form oscillating dipoles. Behind the slab it must cancel $E_{\text{inc}}$ and produce $E_t$. 